Can we trust subjective logic for information fusion?

J. Dezert, A. Tchamova, D. Han, J.M. Tacnet

Can we trust subjective logic for information fusion?

Jean Dezert
The French Aerospace Lab
Chemin de la Hunière
F-91761 Palaiseau, France.
jean.dezert@onera.fr

Albena Tchamova
Institute for IC&Tech.
Bulgarian Acad. of Sci.
Sofia, Bulgaria.
tchamova@bas.bg

Deqiang Han
Inst. of Integrated Automation
Xi'an Jiaotong University
Xi'an, China 710049.
deqhan@gmail.com

Jean-Marc Tacnet
Irstea, UR ETGR
2 Rue de la Papéterie - BP 76
F-38402 St-Martin-d'Hères, France.
jean-marc.tacnet@irstea.fr

***Abstract*—In this paper, we provide a deep examination of the main bases of Subjective Logic (SL) and reveal serious problems with them. A new interesting alternative way for building a normal coarsened basic belief assignment from a refined one is also proposed. The defects in the SL fusion rule and the problems in the link between opinion and Beta probability density functions are also analyzed. Some numerical examples and related analyses are provided to justify our viewpoints.**

***Keywords*—Subjective logic, belief functions, Information fusion.**

I. Introduction

Subjective Logic (SL) was introduced by Jøsang in 1997 [1] as a logic which operates on subjective beliefs about the world. SL is essentially based on the belief functions (BF) introduced in 1976 by Shafer in the development of his Mathematical Theory of Evidence, known as Dempster-Shafer Theory (DST) [2]. According to [3], SL can be seen as an extension of standard (binary) logic and probability calculus. SL introduces new terminologies (like opinions, disbelief, etc) of well-known concepts drawn from DST. Our examination of SL literature reveals that definitions and even notations used in SL have slightly changed over the years that make SL a bit difficult to follow for readers unfamiliar with belief functions basis. The main goal of this paper is to clarify and discuss some bases of SL that appear to us questionable in order to identify the real benefit and innovation of SL with respect to what has been already proposed elsewhere using different notation and terminologies. This paper is organized as follows. Because SL is based on BF, we first recall the basics of BF in Section II. Section III recalls the two main opinion models (simple and "normal") proposed in [1], [4]. We present also in the section III a new alternative "normal" model for building a coarsened basic belief assignment (BBA) based on a strong justification. In Section IV, we discuss the opinion SL fusion rule and in Section V we examine the link between opinion and Beta probability density functions (pdf's). Section VI concludes this paper with general remarks and suggestions.

II. Basics of belief functions

The Belief Functions were introduced in DST [2] to model epistemic uncertainty. We briefly recall in this section the basic definitions of belief functions to make the presentation of Subjective Logic easier to follow. Let Θ be a frame of discernment of a problem under consideration. More precisely, the set $\Theta = \{\theta_1, \theta_2, \ldots, \theta_N\}$ consists of a list of N exhaustive and mutually exclusive elements θ_i, $i = 1, 2, \ldots, N$. Each θ_i represents a possible state related to the problem we want to solve. The assumption of exhaustivity and mutual exclusivity of elements of Θ is classically referred as *Shafer's model* of the frame Θ. A classical[1] basic belief assignment (BBA), also called a belief mass function (or just a mass for short), is a mapping[2] $m_\Theta(.) : 2^\Theta \to [0, 1]$ from the power set[3] of Θ denoted 2^Θ to $[0, 1]$, that verifies [2]:

$$m_\Theta(\emptyset) = 0 \quad \text{and} \quad \sum_{X \in 2^\Theta} m_\Theta(X) = 1 \tag{1}$$

The quantity $m_\Theta(X)$ represents the mass of belief exactly committed to X. An element $X \in 2^\Theta$ is called a focal element if and only if $m_\Theta(X) > 0$. The set $\mathcal{F}(m_\Theta) \triangleq \{X \in 2^\Theta | m_\Theta(X) > 0\}$ of all focal elements of a BBA $m_\Theta(.)$ is called the core of the BBA. A BBA $m_\Theta(.)$ is said Bayesian if its focal elements are singletons of 2^Θ. The vacuous BBA characterizing the total ignorance denoted[4] by $I_t = \theta_1 \cup \theta_2 \cup \ldots \cup \theta_N$ is defined by $m^v_\Theta(.) : 2^\Theta \to [0; 1]$ such that $m^v_\Theta(X) = 0$ if $X \neq \Theta$, and $m^v_\Theta(I_t) = 1$. From any BBA $m_\Theta(.)$, the belief function $Bel_\Theta(.)$ and the plausibility function $Pl_\Theta(.)$ are defined for $\forall X \in 2^\Theta$ as:

$$Bel_\Theta(X) = \sum_{Y \in 2^\Theta | Y \subseteq X} m_\Theta(Y) \tag{2}$$

$$Pl_\Theta(X) = \sum_{Y \in 2^\Theta | X \cap Y \neq \emptyset} m_\Theta(Y) \tag{3}$$

$Bel_\Theta(X)$ represents the whole mass of belief that comes from all subsets of Θ included in X. It is usually interpreted as the lower bound of the probability of X, i.e. $P^{\min}_\Theta(X)$. $Pl_\Theta(X)$ represents the whole mass of belief that comes from all subsets of Θ compatible with X (i.e., those intersecting X). $Pl_\Theta(X)$ is usually interpreted as the upper bound of the probability of X, i.e. $P^{\max}_\Theta(X)$. $Bel_\Theta(.)$ is a sub-additive measure since $\sum_{\theta_i \in \Theta} Bel_\Theta(\theta_i) \leq 1$, whereas $Pl_\Theta(.)$ is super-additive because $\sum_{\theta_i \in \Theta} Pl_\Theta(\theta_i) \geq 1$. $Bel_\Theta(X)$ and $Pl_\Theta(X)$

[1] Some authors prefer to work with unconventional BBAs allowing $m_\Theta(\emptyset) > 0$. We don't use them in this work.

[2] Here we put Θ as subscript to explicitly shows the frame of discernment on which the BBA is referring.

[3] The power set is the set of all subsets of Θ, empty set included.

[4] The set $\{\theta_1, \theta_2, \ldots, \theta_N\}$ and the complete ignorance $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_N$ are both denoted Θ in DST.

defined previously are classically seen [2] as lower and upper bounds of an unknown probability $P_\Theta(.)$, and one has the following inequality satisfied: $\forall X \in 2^\Theta$, $Bel_\Theta(X) \leq P_\Theta(X) \leq Pl_\Theta(X)$. The belief function $Bel_\Theta(.)$ (and the plausibility function $Pl_\Theta(.)$) built from any Bayesian BBA $m_\Theta(.)$ can be interpreted as a (subjective) conditional probability measure provided by a given source of evidence, because if the BBA $m_\Theta(.)$ is *Bayesian* one has [2]: $Bel_\Theta(X) = Pl_\Theta(X) = P_\Theta(X)$.

In DST, the combination (fusion) of several independent sources of evidences is done with Dempster-Shafer[5] (DS) rule of combination, assuming that the sources are not in total conflict[6]. DS combination of two independent BBAs $m^1_\Theta(.)$ and $m^2_\Theta(.)$ is defined by $m^{DS}_\Theta(\emptyset) = 0$, and for all $X \in 2^\Theta \setminus \{\emptyset\}$ by:

$$m^{DS}_\Theta(X) = \frac{1}{K^{DS}} \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = X}} m^1_\Theta(X_1) m^2_\Theta(X_2) \tag{4}$$

with

$$K^{DS} = 1 - \sum_{\substack{X_1,X_2 \in 2^\Theta \\ X_1 \cap X_2 = \emptyset}} m^1_\Theta(X_1) m^2_\Theta(X_2) \tag{5}$$

A discussion on the validity of DS rule and its incompatibility with Bayes fusion rule for combining Bayesian BBAs can be found in [5], [6], [7].

In the next sections, we present and discuss the main concepts and definitions introduced by Jøsang in the development of SL. Our goal is to show that some aspects of SL appear in our standpoint ill-justified and questionable. The three main bases of SL are: 1) the construction of a model for so-called "opinions" from BF, 2) the link made between opinions and Beta probability density functions (pdfs) and 3) a fusion rule to combine opinions provided by different sources. SL proposes also additional propositional operators [4] (simple or normal multiplications, comultiplications,etc) that will not be analyzed nor discussed in the sequel because our main goal here is only to examine the interest of SL for fusion applications.

III. Opinion models in Subjective Logic

SL logic starts with the definitions of two *opinion* models (the simple and normal models) based on Shafer's belief functions introduced in the previous section, and the coarsening of a frame of discernment. Before presenting these models, one must recall three basic notions involved in the definition of opinions: the belief function $b(X)$, the disbelief function $d(X)$ and the uncertainty function $u(X)$.

A. Belief, disbelief and uncertainty

These functions are mathematically defined as follows [1], [3], [4], [8], [9]: $\forall X \in 2^\Theta$

$$b(X) \triangleq \sum_{Y \in 2^\Theta | Y \subseteq X} m_\Theta(Y) \tag{6}$$

$$d(X) \triangleq \sum_{Y \in 2^\Theta | Y \cap X = \emptyset} m_\Theta(Y) \tag{7}$$

$$u(X) \triangleq \sum_{Y \in 2^\Theta | Y \cap X \neq \emptyset \text{ and } Y \not\subseteq X} m_\Theta(Y) \tag{8}$$

These notions appeal few comments[7].

1) In the sequel, we prefer to denote them $b_\Theta(X)$, $d_\Theta(X)$ and $u_\Theta(X)$ to show explicitly the underlying frame we are referring to when manipulating these functions.
2) $b_\Theta(X) = Bel_\Theta(X)$ since they have same definitions.
3) $d_\Theta(X) = Bel_\Theta(\bar{X})$, where $\bar{X}$ denotes the complement of the set X in Θ. The disbelief function $d_\Theta(.)$ corresponds to the doubt function $Dou_\Theta(.)$ already introduced by Shafer in [2], p. 43. More precisely, for all $X \in 2^\Theta$ one has
$$d_\Theta(X) = Dou_\Theta(X) \triangleq Bel_\Theta(\bar{X}) = 1 - Pl_\Theta(X) \tag{9}$$
4) $u_\Theta(X)$ can just be defined more simply by $u_\Theta(X) \triangleq Pl_\Theta(X) - Bel_\Theta(X)$ which is a standard definition to characterize the uncertainty on X in the framework of BF. Because $Pl_\Theta(X) = 1 - Bel_\Theta(\bar{X})$ (see [2] Chap. 2 for proof), one always has $u_\Theta(X) = 1 - Bel_\Theta(\bar{X}) - Bel_\Theta(X)$ as well.
5) For all $X \in 2^\Theta$, $b_\Theta(X) + d_\Theta(X) + u_\Theta(X) = 1$ because $b_\Theta(x) = Bel_\Theta(X)$, $d_\Theta(X) = Bel_\Theta(\bar{X})$ and $u_\Theta(X) = 1 - Bel_\Theta(X) - Bel_\Theta(\bar{X})$.
6) Because $b(X) + d(X) + u(X) = 1$, the characterization of belief of X requires theoretically the knowledge of two values only, typically the belief interval $[Bel_\Theta(X), Pl_\Theta(X)]$, from which other couples of values can be easily drawn, i.e $(Bel_\Theta(X), Bel_\Theta(\bar{X}))$, $(Bel_\Theta(X), u(X))$, $(Bel_\Theta(\bar{X}), u(X))$, etc. In SL, the author works explicitly also with the third redundant component.

B. Atomicity of elements of the frame

The *relative atomicity* of a set X with respect to another set Y (assuming $Y \neq \emptyset$) plays an important role in SL because it is involved in the definition of the pignistic probability (which is also used in SL) as shown in the next subsection. The relative atomicity is defined by [1]

$$a(X|Y) \triangleq \frac{|X \cap Y|}{|Y|} \tag{10}$$

where $|X \cap Y|$ and $|Y|$ are the cardinalities of sets $X \cap Y$ and Y, respectively. Obviously $a(X|Y) \in [0, 1]$ because $(X \cap Y) \subseteq Y$, so that $|X \cap Y| \leq |Y|$ always holds. If Y corresponds to the whole frame $\Theta = \{\theta_1, \ldots, \theta_N\}$ with $N > 1$ mutually

[5]Although the rule has been proposed originally by Dempster, we call it Dempster-Shafer rule because it has been widely promoted by Shafer in DST.

[6]otherwise DS rule is mathematically not defined because of 0/0 indeterminacy.

[7]some of them have been given already by Jøsang himself but are worth to be recalled here.

exclusive *atomic* elements θ_i, then for any set $X \in 2^\Theta$, one has always

$$a(X|\Theta) \triangleq \frac{|X \cap \Theta|}{|\Theta|} = \frac{|X|}{|\Theta|} = \frac{|X|}{N} \quad (11)$$

because $|X \cap \Theta| = |X|$ since $X \subseteq \Theta$. The relative atomicity $a(X|\Theta)$ of X w.r.t[8] Θ is just called *atomicity* of X in SL and it is usually denoted $a(X)$, see [4] for details. We prefer to keep the rigorous notation $a(X|\Theta)$ in the sequel to explicitly denote the underlying frame on which the atomicity is computed. It is worth noting that the knowledge of the numerical value $a(X|\Theta)$ alone does not bring a very useful information on X if we don't know jointly the cardinality of Θ. For example, from the given numerical value $a(X|\Theta) = 0.5$, we can only infer that the size (cardinality) of X is only half of the size of the frame Θ. Without knowing the cardinality of Θ itself, the cardinality of X cannot be known from the given value of $a(X|\Theta)$.

C. Pignistic probability

The pignistic[9] probability measure, denoted by $BetP_\Theta(.)$, is a particular subjective probability measure consistent with belief intervals computed from any BBA $m_\Theta(.)$ defined on 2^Θ, i.e. such that $Bel_\Theta(X) \leq BetP_\Theta(X) \leq Pl_\Theta(X)$ for all $X \subseteq \Theta$. $BetP_\Theta(.)$ was proposed and coined by Smets and Kennes in [10] and its justification is based on the principle of insufficient reason defended by the authors. It has been renamed *probability expectation*, and denoted by $E[.]$ in SL [1], [8]. In fact, there exists many other ways to construct (subjective) probability measures $P_\Theta(.)$ that are also consistent with belief intervals for approximating any BBA $m_\Theta(.)$ as already reported in [11], Chap. 3. All methods of construction of $P_\Theta(.)$ are based on different justifications and there is no consensus on which one is the most efficient and useful one, even if the pignistic probability is very often adopted in practice. In our opinion, there is no unique "expected probability", and that is why it seems more judicious to keep the original name (pignistic probability) in the sequel in order to refer precisely to the probability transformation used. The pignistic probability $BetP_\Theta(.)$ is defined[10] by:

$$BetP_\Theta(X) = \sum_{Y \in 2^\Theta \setminus \{\emptyset\}} a(X/Y) m_\Theta(Y) \quad (12)$$

$BetP_\Theta(.)$ satisfies all three Kolmogorov probabilities axioms and its consequences; in particular one always has $BetP_\Theta(X \cup Y) = BetP_\Theta(X) + BetP_\Theta(Y)$ if $X \cap Y = \emptyset$, $X, Y \subseteq \Theta$.

Remark 1: According to [3], p. 5, the "*belief functions can only be used to estimate probability values and not to set bounds, because the probability of a real event can never be determined with certainty, and neither can upper and lower bounds to it.*" This interpretation is unconventional and disputable because the definitions of $Bel_\Theta(.)$ and $Pl_\Theta(.)$ functions from a BBA $m_\Theta(.)$ allow to compute mathematically the lower and upper bounds of a consistent subjective probability measure $P_\Theta(X)$ for all $X \in 2^\Theta$ as explained by Shafer in [2] as soon as a BBA $m_\Theta(.)$ is precisely known or given.

Focal Elem.	A	B	C	$A \cup B$	$A \cup C$	$B \cup C$	Θ
$m_\Theta(.)$	0.10	0.15	0.05	0.10	0.10	0.20	0.30
$b(.)$	0.10	0.15	0.05	0.35	0.25	0.40	1
$d(.)$	0.40	0.25	0.35	0.05	0.15	0.10	0
$u(.)$	0.50	0.60	0.60	0.60	0.60	0.50	0
$a(.\|\Theta)$	1/3	1/3	1/3	2/3	2/3	2/3	1
$BetP_\Theta(.)$	0.30	0.40	0.30	0.70	0.60	0.70	1

TABLE I. COMPUTATIONS OF $b(X)$, $d(X)$, $u(X)$, $a(X|\Theta)$ AND $BetP_\Theta(X)$.

Example 1: Let us consider the frame $\Theta = \{A, B, C\}$ with Shafer's model. Table I gives the values of $b(x) = Bel_\Theta(X)$, $d(X) = Bel_\Theta(\bar{X})$, $u(X) = 1 - Bel_\Theta(X) - Bel_\Theta(\bar{X})$, $a(X|\Theta)$ and $BetP_\Theta(X)$ for all elements $X \in 2^\Theta \setminus \{\emptyset\}$ for the BBA $m_\Theta(.)$ given in the second row of the Table.

D. Subjective Opinion

Originally in [1], an opinion about X, denoted ω_X is an ordered triple of real values $(b(X), d(X), u(X))$ in $[0, 1]^3$

$$\omega_X \triangleq (b(X), d(X), u(X)) \quad (13)$$

with the constraint $b(X) + d(X) + u(X) = 1$ where $b(X)$ is the belief component of the opinion ω_X, $d(X)$ is its disbelief, and $u(X)$ is its uncertainty. It is clear that ω_X as given by Eq. (13) corresponds exactly to a simple BBA $m_{\Theta_X}(.) = (m_{\Theta_X}(X), m_{\Theta_X}(\bar{X}), m_{\Theta_X}(X \cup \bar{X}))$ defined on the power-set of the 2D frame $\Theta_X \triangleq \{X, \bar{X}\}$, and it is a trivial instance of DST, see [1], p. 3. Actually for technical reasons that will appear clear in the sequel, we prefer to adopt a more rigorous notation and introduce explicitly the frame Θ_X in the notation. Hence, we denote an opinion as:

$$\omega_{\Theta_X}(X) \triangleq (b_{\Theta_X}(X), d_{\Theta_X}(X), u_{\Theta_X}(X)) \quad (14)$$

Because $m_{\Theta_X}(X) + m_{\Theta_X}(\bar{X}) + m_{\Theta_X}(X \cup \bar{X}) = 1$, one component is redundant, and in fact an opinion $\omega_{\Theta_X}(X)$ is mathematically equivalent to the knowledge of only two components expressed as a belief interval, i.e. $\omega_{\Theta_X}(X) \equiv [Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)]$ with:

$$b_{\Theta_X}(X) \triangleq Bel_{\Theta_X}(X) = m_{\Theta_X}(X) \quad (15)$$

$$d_{\Theta_X}(X) \triangleq Bel_{\Theta_X}(\bar{X}) = 1 - Pl_{\Theta_X}(X) = m_{\Theta_X}(\bar{X}) \quad (16)$$

$$u_{\Theta_X}(X) \triangleq Pl_{\Theta_X}(X) - Bel_{\Theta_X}(X) = m_{\Theta_X}(X \cup \bar{X}) \quad (17)$$

$m_{\Theta_X}(X)$ can be interpreted as the percentage of truth of the proposition X = "The solution is in the set X", $m_{\Theta_X}(\bar{X})$ as the percentage of falsehood of the proposition X, and $m_{\Theta_X}(X \cup \bar{X})$ as the percentage of subjective uncertainty about the proposition X. By construction of belief functions, one always has:

$$\begin{aligned} u_{\Theta_X}(X) &= Pl_{\Theta_X}(X) - Bel_{\Theta_X}(X) \\ &= (1 - Bel_{\Theta_X}(\bar{X})) - (1 - Pl_{\Theta_X}(\bar{X})) \\ &= Pl_{\Theta_X}(\bar{X}) - Bel_{\Theta_X}(\bar{X}) = u_{\Theta_X}(\bar{X}) \end{aligned} \quad (18)$$

It is worth noting that the knowledge of the belief interval $[Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)]$, or equivalently the knowledge of

[8] with respect to;

[9] *pignistic* epithet comes from *pignus* in Latin which means *bet*. That is why $BetP$ notation is used to refer to this betting commitment probability.

[10] when working with normal BBA, i.e. when $m_\Theta(\emptyset) = 0$.

$m_{\Theta_X}(.)$, doesn't require the knowledge of cardinalities of X, $\bar{X}$, nor Θ_X. The knowledge of these cardinalities is necessary only to estimate a compatible pignistic probability measure $BetP_{\Theta_X}(X) \in [Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)]$ because it needs the relative atomicities for its derivation. Moreover, any ordered triple $(b(X), d(X), u(X))$ in $[0,1]^3$ with $b(X)+d(X)+u(X)=1$ can always be identified directly as a simple BBA $m_{\Theta_X}(.)$ according to (15)–(17), without the necessity of referring to a refined[11] BBA $m_\Theta(.)$ through (6)–(8). In practice, the belief assessments are preferentially expressed by human experts directly with simple (coarsened) BBAs rather than with refined BBA because the latter are much more difficult to establish. Therefore the definition of an opinion model based on a refined BBA according to (6)–(8), although receivable, is not fundamental in the strict definition of an opinion since it should be directly given by (15)–(17).

Example 2: let us consider the 2D-frame $\Theta_X \triangleq \{X, \bar{X}\}$ with the BBA $m_{\Theta_X}(.)$ given by:

$$\begin{cases} m_{\Theta_X}(X) = 0.10 \\ m_{\Theta_X}(\bar{X}) = 0.40 \\ m_{\Theta_X}(X \cup \bar{X}) = 0.50 \end{cases}$$

which corresponds to the belief intervals (opinions) $\omega_{\Theta_X}(X) \equiv [Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)] = [0.10, 0.60]$ and $\omega_{\Theta_X}(\bar{X}) \equiv [Bel_{\Theta_X}(\bar{X}), Pl_{\Theta_X}(\bar{X})] = [0.40, 0.90]$. Without extra assumptions on $|X|$ and $|\bar{X}|$ there is no way to compute $BetP_{\Theta_X}(X)$. One can nevertheless apply the pignistic transformation of $m_{\Theta_X}(.)$ with different assumptions to get a compatible probability measure with the belief intervals. For example,

• By taking the pignistic transformation and making the naive assumption (by default) $|X| = |\bar{X}|$, so that $a(X|\Theta_X) = \frac{|X|}{|\Theta_X|} = \frac{|X|}{|X|+|\bar{X}|} = 1/2 = a(\bar{X}|\Theta_X)$, one will get:

$$\begin{cases} BetP_{\Theta_X}(X) = m_{\Theta_X}(X) + \frac{1}{2} m_{\Theta_X}(X \cup \bar{X}) = 0.35 \\ BetP_{\Theta_X}(\bar{X}) = m_{\Theta_X}(\bar{X}) + \frac{1}{2} m_{\Theta_X}(X \cup \bar{X}) = 0.65 \end{cases}$$

Note that this probability measure doesn't require the full knowledge of the values of $|X|$ and $|\bar{X}|$, but only that the equality $|X| = |\bar{X}|$ holds.

• By taking the pignistic transformation and assuming some extra knowledge of cardinalities $|X|$ and $|\bar{X}|$, say for example let assume $|X| = 1$ and $|\bar{X}| = 2$ so that $a(X|\Theta_X) = \frac{|X|}{|\Theta_X|} = \frac{|X|}{|X|+|\bar{X}|} = 1/3$ and $a(\bar{X}|\Theta_X) = 2/3$, one will get:

$$\begin{cases} BetP_{\Theta_X}(X) = m_{\Theta_X}(X) + \frac{1}{3} m_{\Theta_X}(X \cup \bar{X}) \approx 0.267 \\ BetP_{\Theta_X}(\bar{X}) = m_{\Theta_X}(\bar{X}) + \frac{2}{3} m_{\Theta_X}(X \cup \bar{X}) \approx 0.733 \end{cases}$$

E. Building opinions from a refined BBA

If one has a refined BBA $m_\Theta(.)$ defined on the power-set of a frame Θ, it is always possible to compute the belief interval $[Bel_\Theta(X), Pl_\Theta(X)]$ for all $X \in 2^\Theta$ according to (2)-(3). Reciprocally, from all belief $Bel_\Theta(X)$ values (or plausibility values $Pl_\Theta(X)$), one can always compute the BBA $m_\Theta(.)$ thanks to Möbius transform [2]. Moreover, if one has $m_\Theta(.)$ and if one knows the cardinality of the element X for all $X \in 2^\Theta$, one can easily compute the pignistic probability $BetP_\Theta(X)$ thanks to (12).

From the knowledge of a given BBA $m_\Theta(.)$, how to build a simpler (coarsened) BBA $m_{\Theta_X}(.)$ (called "opinion" in SL) for any chosen $X \in 2^\Theta$? Several methods are possible for doing this. We briefly present them, and propose a new one based on a strong mathematical justification.

• **Simple model** [12]: the knowledge of the belief interval $[Bel_\Theta(X), Pl_\Theta(X)]$ allows to define very simply $m_{\Theta_X}(.)$ on the coarsened 2D-frame[12] $\Theta_X \triangleq \{X, \bar{X}\}$ by taking trivially:

$$\begin{cases} m_{\Theta_X}(X) & = Bel_\Theta(X) \\ m_{\Theta_X}(\bar{X}) & = 1 - Pl_\Theta(X) = Bel_\Theta(\bar{X}) \\ m_{\Theta_X}(X \cup \bar{X}) & = Pl_\Theta(X) - Bel_\Theta(X) \\ & = 1 - Bel_\Theta(X) - Bel_\Theta(\bar{X}) \end{cases} \tag{19}$$

In 2001, Jøsang did reconsider this simple model by taking also into account the atomicity of element X in the extended definition of an opinion. Such extension was probably motived by the main role taken by the atomicities in the computation of the pignistic probabilities. The (extended) opinion definition used in SL from 2001 is as follows [3]

$$\omega_X \triangleq (b(X), d(X), u(X), a(x)) \tag{20}$$

In our point of view, even if atomicity is an important information about the structure of the frame, an opinion including such information should be expressed more rigorously by one of the two formulae depending on which underlying frame the BBA we are directly referring

$$\omega_\Theta(X) \triangleq ([Bel_\Theta(X), Pl_\Theta(X)], a(X|\Theta)) \tag{21}$$

$$\omega_{\Theta_X}(X) \triangleq ([Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)], a(X|\Theta_X)) \tag{22}$$

It is worth noting that imposing $[Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)] = [Bel_\Theta(X), Pl_\Theta(X)]$ and $a(X|\Theta_X) = a(X|\Theta)$ does not help to get $BetP_{\Theta_X}(X) = BetP_\Theta(X)$ when working with $m_\Theta(.)$ or when working with the coarsened BBA $m_{\Theta_X}(.)$ built with (19). This is not surprising at all because on the one hand of the distinct granularities of the frames Θ and Θ_X and on the other hand of the probabilistic formula (12) itself. This remark is perfectly illustrated in the following example.

Example 3: Let us consider $\Theta = \{A, B, C\}$ with Shafer's model and the BBA $m_\Theta(.)$ as given in Table I, and let's take $X = A$ so that $m_{\Theta_X}(.)$ corresponds exactly to the BBA given in example 2 according to (19). As already shown in example 1, one has with $m_\Theta(.)$, $BetP_\Theta(X = A) = 0.30$ and $BetP_\Theta(\bar{X} = B \cup C) = BetP_\Theta(B) + BetP_\Theta(C) = 0.70$. If we assume same exact atomicity values $a(X|\Theta_X) = a(X|\Theta) = 1/3$ and $a(\bar{X}|\Theta_X) = a(\bar{X}|\Theta) = 2/3$, we obtain with $m_{\Theta_X}(.)$ given in example 2 $BetP_{\Theta_X}(X = A) = 0.267$ and $BetP_{\Theta_X}(\bar{X}) = 0.733$. We clearly get different pignistic probability values with $m_\Theta(.)$ and with $m_{\Theta_X}(.)$, even if

[11] We call the BBA $m_\Theta(.)$ refined because $|\Theta| > |\Theta_X|$.

[12] also called the *focused frame* in [8].

we use the same exact atomicities in BetP formulae and if we impose $[Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)] = [Bel_\Theta(X), Pl_\Theta(X)]$, which is perfectly normal.

In order to circumvent this inherent incompatibility problem, Jøsang proposed to redefine the "atomicity" of X with respect to Θ_X from the knowledge of the pignistic probability $BetP_\Theta(.)$ computed from the BBA $m_\Theta(.)$. More precisely, to get $BetP_{\Theta_X}(X) = m_{\Theta_X}(X) + \tilde{a}(X|\Theta_X)m_{\Theta_X}(X \cup \bar{X}) = BetP_\Theta(X)$ when imposing $[Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)] = [Bel_\Theta(X), Pl_\Theta(X)]$, one must trivially take

$$\tilde{a}(X|\Theta_X) \triangleq (BetP_\Theta(X) - m_{\Theta_X}(X))/m_{\Theta_X}(X \cup \bar{X}) \quad (23)$$

with assuming[13] $m_{\Theta_X}(X \cup \bar{X}) > 0$.

Remark 2: This redefinition of the relative atomicity of the element X of the coarsened frame Θ_X doesn't of course reflect the true relative atomicity of X, but it defines a (a posteriori[14]) redistribution factor of the mass of uncertainty $m_{\Theta_X}(X \cup \bar{X})$ to $m_{\Theta_X}(X)$ for calculating a subjective probability measure $P_{\Theta_X}(X)$. In fact, the knowledge of $P_{\Theta_X}(X)$ or $\tilde{a}(X|\Theta_X)$ are strictly equivalent as soon as the BBA $m_{\Theta_X}(.)$ is given. Therefore, the (extended) opinion defined by (20) could be also replaced more judiciously by:

$$\begin{aligned}\omega_{\Theta_X}(X) &\triangleq (m_{\Theta_X}(X), m_{\Theta_X}(\bar{X}), m_{\Theta_X}(X \cup \bar{X}), P_{\Theta_X}(X)) \\ &\equiv ([Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)], P_{\Theta_X}(X))\end{aligned}$$

where $P_{\Theta_X}(X)$ could be any subjective probability measure compatible with the belief interval $[Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)]$, or eventually computed from $m_\Theta(.)$, if known.

• **Jøsang's "normal" model:** Later in 2003, Jøsang did propose in [4], [14], [15] another model to build $m_{\Theta_X}(.)$ from the belief interval $[Bel_\Theta(X), Pl_\Theta(X)]$, the atomicity $a(X|\Theta)$ and from $BetP_\Theta(X)$. The motivation was to preserve the same atomicities values, i.e. $a(X|\Theta_X) = a(X|\Theta)$ in the model and to get same pignistic probabilities value $BetP_{\Theta_X}(X) = BetP_\Theta(X)$, and that is why it has been called the "normal" model. This model is given by the following formulae – For $BetP_\Theta(X) \geq Bel_\Theta(X) + a(X|\Theta)u_\Theta(X)$

$$\begin{cases} m_{\Theta_X}(X) \triangleq Bel_\Theta(X) + \frac{BetP_\Theta(X) - Bel_\Theta(X) - a(X|\Theta)u_\Theta(X)}{1 - a(X|\Theta)} \\ m_{\Theta_X}(\bar{X}) \triangleq d_\Theta(X) = Bel_\Theta(\bar{X}) \\ m_{\Theta_X}(X \cup \bar{X}) \triangleq u_\Theta(X) - \frac{BetP_\Theta(X) - Bel_\Theta(X) - a(X|\Theta)u_\Theta(X)}{1 - a(X|\Theta)} \\ a(X|\Theta_X) \triangleq a(X|\Theta) \end{cases} \quad (24)$$

– For $BetP_\Theta(X) < Bel_\Theta(X) + a(X|\Theta)u_\Theta(X)$

$$\begin{cases} m_{\Theta_X}(X) \triangleq Bel_\Theta(X) \\ m_{\Theta_X}(\bar{X}) \triangleq Bel_\Theta(\bar{X}) + \frac{Bel_\Theta(X) + a(X|\Theta)u_\Theta(X) - BetP_\Theta(X)}{a(X|\Theta)} \\ m_{\Theta_X}(X \cup \bar{X}) \triangleq u_\Theta(X) - \frac{Bel_\Theta(X) + a(X|\Theta)u_\Theta(X) - BetP_\Theta(X)}{a(X|\Theta)} \\ a(X|\Theta_X) \triangleq a(X|\Theta) \end{cases} \quad (25)$$

Remark 3: This mathematical "normal model" has never been seriously justified in SL literature, and in fact we don't see solid arguments for supporting it, mainly because a better (simpler) and more strongly justified model can be easily established as it will be shown in the next paragraph.

• **A new and better "normal" model:** this new model derives from one of Shannon's entropy properties which states that entropy decreases when the size of the probability space decreases because the number of possible states taken by the random variable (governed by the underlying probability measure) diminishes. The basic idea to build a better new "normal" model is to use this property to quantify precisely the gain in the reduction of uncertainty that one has to consider when working in the coarsened frame Θ_X. Before going further in the formulation of our new "normal" model, one needs to first recall few bases about Shannon entropy.

Shannon entropy [16], usually expressed in bits (binary digits), of a probability measure $P_N(X) \triangleq [p_1, p_2, \ldots, p_N]$ of a random variable X over a discrete finite set $\Theta_N = \{\theta_1, \ldots, \theta_N\}$ is defined by[15]

$$H(P_N) = H_N(p_1, p_2, \ldots, p_N) \triangleq -\sum_{i=1}^{N} p_i \log_2 p_i \quad (26)$$

where $p_i \triangleq P(X = \theta_i)$. The entropy $H(P_N)$ measures the degree of randomness carried by a given distribution of probabilities of X over the set Θ_N of cardinality N. The entropy is minimal if there exists a $\theta_i \in \Theta$ such that $P(X = \theta_i) = 1$ so that $H_N^{\min} = 0$. It is maximal when the distribution of probabilities of X is uniform on Θ_N, that is when $P(X = \theta_i) = 1/N$, $i = 1, 2, \ldots, N$. In that case, $H_N^{\max} = \log_2(N)$. Therefore, $H(P_N) \in [0, \log_2(N)]$ when expressed in bits. An interesting recursive property of Shannon entropy is that [16], [17] if $p_1 + p_2 > 0$

$$\begin{aligned} H_N(p_1, p_2, \ldots, p_N) &= H_{N-1}(p_1 + p_2, p_3 \ldots, p_N) \\ &+ (p_1 + p_2) H_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) \end{aligned} \quad (27)$$

The quantity $H_{N-1}(p_1 + p_2, p_3 \ldots, p_N)$ represents the entropy of the probability measure $P_{N-1}(X) \triangleq [p_1 + p_2, p_3 \ldots, p_N]$ of the random variable X over the discrete coarsened finite set $\Theta_{N-1} \triangleq \{\theta_1 \cup \theta_2, \theta_3, \ldots, \theta_N\}$ having now $N - 1$ elements. The term $(p_1 + p_2)H_2(\frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}) \geq 0$ reflects the increase of entropy due to the refinement process when splitting $\theta_1 \cup \theta_2$ into two separate states θ_1 and θ_2. The increase of entropy is normal because the refinement process generates more uncertainty on the choice to make due to the increase of granularity of the frame. We define the entropy reduction factor $\alpha_N(N, N - 1) \in [0, 1]$ by:

$$\alpha \triangleq \frac{H_{N-1}}{H_N} \quad (28)$$

So let's now consider a BBA $m_\Theta(.)$ defined on 2^Θ and a subjective probability measure, say $BetP_\Theta(.)$ with the pignistic transformation, over the frame Θ of cardinality N. Based on $BetP_\Theta(.)$, we are able to compute its corresponding Shannon entropy $H_N(BetP_\Theta)$, that we just denote as H_{fine} to specify

[13] if $m_{\Theta_X}(X \cup \bar{X}) = 0$, then it means from (19) that $Bel_\Theta(X) = Pl_\Theta(X) = BetP_\Theta(X) = BetP_{\Theta_X}(X)$.

[14] assuming the given knowledge of $BetP_\Theta(X)$.

[15] with convention $0 \log_2 0 = 0$.

that it has been computed from the probability defined over a refined frame. Let's consider an element $X \in 2^\Theta$ and its corresponding coarsened frame $\Theta_X = \{X, \bar{X}\}$. We are able from the set of values $BetP_\Theta(\theta_i)$, $i = 1, 2, \ldots, N$ to compute also the *coarsened probability* measure by taking $BetP_\Theta(X) = \sum_{\theta_i \in X} BetP_\Theta(\theta_i)$ and $BetP_\Theta(\bar{X}) = 1 - BetP_\Theta(X)$. This subjective (coarsened) probability measure yields a new (smaller) Shannon entropy value denoted H_{coarse} because of the coarsening effect as shown through (27). Hence the entropy reduction factor $\alpha = H_{coarse}/H_{fine}$ is easily obtained. This reduction factor will be used to reduce the uncertainty degree in our new normal model by trivially taking:

$$u_{\Theta_X}(X) \triangleq \alpha \cdot u_\Theta(X) = \frac{H_{coarse}}{H_{fine}} \cdot (Pl_\Theta(X) - Bel_\Theta(X)) \quad (29)$$

$u_{\Theta_X}(X) \in [0, 1]$, because $\alpha \in [0, 1]$ and $u_\Theta(X) \in [0, 1]$.

From any BBA $m_{\Theta_X}(.)$ defined in $\Theta_X = \{X, \bar{X}\}$, it is always possible to compute the pignistic probabilities $BetP_{\Theta_X}(X)$ and $BetP_{\Theta_X}(\bar{X}) = 1 - BetP_{\Theta_X}(X)$ by the probability transformation given in (12). More precisely, by

$$\begin{cases} BetP_{\Theta_X}(X) = m_{\Theta_X}(X) + a(X|\Theta_X) m_{\Theta_X}(X \cup \bar{X}) \\ BetP_{\Theta_X}(\bar{X}) = m_{\Theta_X}(\bar{X}) + a(\bar{X}|\Theta_X) m_{\Theta_X}(X \cup \bar{X}) \end{cases} \quad (30)$$

Conversely, from any given set of consistent[16] probabilities $(BetP_{\Theta_X}(X), BetP_{\Theta_X}(\bar{X}))$ and any chosen value $m_{\Theta_X}(X \cup \bar{X})$ in $[0, 1]$, we can always establish a well-defined BBA by computing $m_{\Theta_X}(X)$ and $m_{\Theta_X}(\bar{X})$ from (30) by taking

$$\begin{cases} m_{\Theta_X}(X) = BetP_{\Theta_X}(X) - a(X|\Theta_X) m_{\Theta_X}(X \cup \bar{X}) \\ m_{\Theta_X}(\bar{X}) = BetP_{\Theta_X}(\bar{X}) - a(\bar{X}|\Theta_X) m_{\Theta_X}(X \cup \bar{X}) \end{cases} \quad (31)$$

In particular, in choosing $(BetP_{\Theta_X}(X), BetP_{\Theta_X}(\bar{X})) = (BetP_\Theta(X), BetP_\Theta(\bar{X}))$ and $m_{\Theta_X}(X \cup \bar{X}) = u_{\Theta_X}(X) = \alpha \cdot u_\Theta(X)$, because $(BetP_\Theta(X), BetP_\Theta(\bar{X}))$ is a consistent couple of probabilities and $\alpha \cdot u_\Theta(X) \in [0, 1]$. Since $BetP_\Theta(\bar{X}) = 1 - BetP_\Theta(X)$ and $a(X|\Theta_X) = a(X|\Theta)$ and $a(\bar{X}|\Theta_X) = a(\bar{X}|\Theta) = 1 - a(X|\Theta)$, we finally obtain thanks to (31) the explicit formulae of our new "normal" model for the well-defined coarsened BBA $m_{\Theta_X}(.)$ as follows:

$$\begin{cases} m_{\Theta_X}(X) \triangleq BetP_\Theta(X) - a(X|\Theta_X) \cdot \alpha \cdot (Pl_\Theta(X) - Bel_\Theta(X)) \\ m_{\Theta_X}(\bar{X}) \triangleq (1 - BetP_\Theta(X)) \\ \qquad\qquad - (1 - a(X|\Theta_X)) \cdot \alpha \cdot (Pl_\Theta(X) - Bel_\Theta(X)) \\ m_{\Theta_X}(X \cup \bar{X}) \triangleq \alpha \cdot (Pl_\Theta(X) - Bel_\Theta(X)) \\ a(X|\Theta_X) = a(X|\Theta) \end{cases} \quad (32)$$

where the entropy reduction factor α is given by $\alpha = H(BetP_{\Theta_X})/H(BetP_\Theta)$ with

$$BetP_{\Theta_X} = [\sum_{\theta_i \in X} BetP_\Theta(\theta_i), 1 - \sum_{\theta_i \in X} BetP_\Theta(\theta_i)] \quad (33)$$

$$BetP_\Theta = [BetP_\Theta(\theta_1), \ldots, BetP_\Theta(\theta_N)] \quad (34)$$

Remark 4: it is worth noting that the factor $\alpha \in [0, 1]$ is independent of the logarithm base used to define and compute the entropies involved in (28) (instead of $\log_2$, it could be $\log_{10}$, $\ln$, $\log_b$ with $b > 0$ different of 1), because for any real positive numbers a and b different of 1 there always exists a constant k such that $\log_b = k \log_a$ which is given by $k = 1/\log_a(b)$. Therefore, our model doesn't depend on the choice of the logarithm base involved in the entropy definition. It is also easy to verify that $BetP_{\Theta_X}(X) = m_{\Theta_X}(X) + a(X|\Theta_X) m_{\Theta_X}(X \cup \bar{X})$ is equal to $BetP_\Theta(X)$ by using formula (32). In summary, this new "normal" model has in our point of view a better construction and strongly justification than Jøsang's "normal" model given in (24)–(25), even if its practical interest remains to be shown for real applications.

Example 4: let us consider $\Theta = \{A, B, C\}$ with Shafer's model and the BBA $m_\Theta(.)$, atomicities and BetP values as given in Table I, and let's take $X = A$ so that $\Theta_X = \{X = A, \bar{X} = B \cup C\}$. Because $BetP_\Theta(X = A) = 0.30$ is greater than $Bel_\Theta(X) + a(X|\Theta) u_\Theta(X) = 0.10 + (1/3)0.50 \approx 0.266$, we use the formulae in (24) to find the normal Jøsang's model. Based on (24), one will get:

$$\begin{cases} m_{\Theta_X}(X) = 0.10 + \frac{0.30 - 0.1 - (1/3) \cdot 0.5}{1 - (1/3)} = 0.15 \\ m_{\Theta_X}(\bar{X}) = Bel_\Theta(B \cup C) = 0.40 \\ m_{\Theta_X}(X \cup \bar{X}) = 0.50 - \frac{0.30 - 0.1 - (1/3) \cdot 0.5}{1 - (1/3)} = 0.45 \\ a(X|\Theta_X) = a(X|\Theta) = 1/3 \end{cases} \quad (35)$$

Based on $BetP_\Theta(.)$, one has $H_{fine} \approx 1.57095$ and $H_{coarse} \approx 0.88129$ so that $\alpha \approx 0.56099$. Applying the new normal model formulae given in (32) we obtain now:

$$\begin{cases} m_{\Theta_X}(X) = 0.30 - (1/3) \cdot \alpha \cdot (0.60 - 0.10) \approx 0.20650 \\ m_{\Theta_X}(\bar{X}) = 0.70 - (2/3) \cdot \alpha \cdot (0.60 - 0.10) \approx 0.51300 \\ m_{\Theta_X}(X \cup \bar{X}) = 0.56099 \cdot (0.60 - 0.10) \approx 0.28050 \\ a(X|\Theta_X) = a(X|\Theta) = 1/3 \end{cases} \quad (36)$$

With our new model, we get a reduction of uncertainty that directly reflects the decrease of Shannon entropy of the pignistic probability measure when working on a coarsened frame. Such reduction level of uncertainty is better justified than the reduction proposed in Jøsang's "ad-hoc normal" model.

IV. Fusion of Opinions in Subjective Logic

In this section, we examine the fusion rules proposed in SL to combine two opinions given by two distinct sources of evidence related to the same frame Θ_X. For notation convenience, in the sequel we denote SL opinions $\omega_X^1 = (b_X^1, d_X^1, u_X^1)$ and $\omega_X^2 = (b_X^2, d_X^2, u_X^2)$ using the conventional notation, that is $\omega_X^1 \equiv m_1(.) = (m_1(X), m_1(\bar{X}), m_1(X \cup \bar{X}))$ and $\omega_X^2 \equiv m_2(.) = (m_2(X), m_2(\bar{X}), m_2(X \cup \bar{X}))$.

A. SL Consensus rule

In 1997, Jøsang did propose in [1] the following associative consensus rule to combine two independent BBAs $m_1(.)$ and

[16] such that $(BetP_{\Theta_X}(X), BetP_{\Theta_X}(\bar{X})) \in [0, 1]^2$, with $BetP_{\Theta_X}(X) + BetP_{\Theta_X}(\bar{X}) = 1$.

$m_2(.)$ defined over the same frame $\Theta_X = \{X, \bar{X}\}$

$$\begin{cases} m^{1,2}(X) = \frac{1}{K} \cdot (m_1(X)m_2(X \cup \bar{X}) + m_2(X)m_1(X \cup \bar{X})) \\ m^{1,2}(\bar{X}) = \frac{1}{K} \cdot (m_1(\bar{X})m_2(X \cup \bar{X}) + m_2(\bar{X})m_1(X \cup \bar{X})) \\ m^{1,2}(X \cup \bar{X}) = \frac{1}{K} \cdot m_1(X \cup \bar{X})m_2(X \cup \bar{X}) \end{cases} \quad (37)$$

where the normalization constant K is given by:

$$K = m_1(X\cup\bar{X}) + m_2(X\cup\bar{X}) - m_1(X\cup\bar{X})m_2(X\cup\bar{X}) \quad (38)$$

The combined basic belief assignment $m^{1,2}(.) \triangleq (m^{1,2}(X), m^{1,2}(\bar{X}), m^{1,2}(X \cup \bar{X}))$ is called the consensus opinion according to Jøsang's definition.

Contrariwise to most rules developed so far in the BF community, the rule defined by (37) is not able to combine mathematically Bayesian BBAs because the formula (37) doesn't work when $m_1(X \cup \bar{X}) = m_2(X \cup \bar{X}) = 0$. Later in 2002, the author proposed an adaptation in [13] to handle the Bayesian BBAs case separately when $K = 0$ which was mathematically erroneous[17]. In fact, the Bayesian case which cannot be directly managed by this rule is not a very serious problem because it is easy to prove by taking $m_1(X \cup \bar{X}) = m_2(X \cup \bar{X}) = \epsilon$ with $\epsilon > 0$ close to zero that the consensus rule reduces to the simple (non associative) averaging rule when taking $\epsilon = 0$. The proof is left to the reader.

A much more serious problem with this SL consensus rule is its illogical construction because it doesn't manage correctly the pure conjunctive consensus expressed by the products $m_1(X)m_2(X)$ and $m_1(\bar{X})m_2(\bar{X})$. Indeed, in this rule $m_1(X)m_2(X)$ is not directly committed to $m^{1,2}(X)$, and $m_2(\bar{X})m_2(\bar{X})$ is not directly committed to $m^{1,2}(\bar{X})$ either, which seems a very unreasonable way to process the information given by the sources. Why? because in one hand the products $m_1(X)m_2(X \cup \bar{X})$ (as well as $m_2(X)m_1(X \cup \bar{X})$) directly contribute in the combined mass of X because $X \cap (X \cup \bar{X}) = X$ which makes sense from the conjunctive standpoint, whereas $m_1(X)m_2(X)$ does not contribute although one has the conjunction $X \cap X = X$. We don't see any serious and logical reason for making a difference in the redistribution of these products because they refer to the same element X by conjunctive operation. This appears totally illogical and so we have very serious doubts on the validity and on the interest of such rule for combing simple BBAs as proposed in SL. In fact in this rule, the products $m_1(X)m_2(X)$ and $m_2(\bar{X})m_2(\bar{X})$ are managed in the same manner as if they were true conflicting terms like $m_1(\bar{X})m_2(X)$ and $m_1(X)m_2(\bar{X})$ through the normalization constant K, whereas they clearly correspond to pure conjunctive consensus. Our previous claim is directly supported by the fact that one can easily prove

$$K = K^{DS} - m_1(X)m_2(X) - m_1(\bar{X})m_2(\bar{X}) \quad (39)$$

In 2001, Jøsang proposed also in [3] the same consensus rule with an extension by adding a separate term to compute the relative combined atomicity of X for reflecting the fact that sources may have different estimates of relative atomicities of X in Θ_X. The need for the computation of this relative combined atomicity of X is however disputable. Why? Because if sources have different interpretations (estimations) of relative atomicities of X, it means that the structure of Θ_X is different for each source. In this case, the direct fusion of the BBAs doesn't make sense, because the fusion must apply only in a same common frame. The relative atomicity depends only of the structure of the frame as shown in (10), and not of the sources providing the BBAs.

In [3], the author justifies the SL consensus rule from another consensus operator defined from two Beta pdf's mapped with BBAs $m_1(.)$ and $m_2(.)$. Because this SL consensus rule is very disputable as we have explained previously, it is very probable that the consensus operator defined for the two Beta pdf's is flawed, or the mapping between Beta pdf and BBA, or both. This point is discussed in the next section.

V. Link between opinion and Beta pdf

Let us consider a BBA defined on the power set of the 2D frame $\Theta_X = \{X, \bar{X}\}$ and given by $m_{\Theta_X}(.) = [m_{\Theta_X}(X), m_{\Theta_X}(\bar{X}), m_{\Theta_X}(X \cup \bar{X})]$. This BBA can be interpreted as an imprecise subjective probability measure $P_{\Theta_X}(.)$ verifying $P_{\Theta_X}(X) + P_{\Theta_X}(\bar{X}) = 1$ and bounded by the belief intervals $P_{\Theta_X}(X) \in [Bel_{\Theta_X}(X), Pl_{\Theta_X}(X)]$ and $P_{\Theta_X}(\bar{X}) \in [Bel_{\Theta_X}(\bar{X}), Pl_{\Theta_X}(\bar{X})]$ with $Bel_{\Theta_X}(\bar{X}) = 1 - Pl_{\Theta_X}(X)$ and $Pl_{\Theta_X}(\bar{X}) = 1 - Bel_{\Theta_X}(X)$. So, the unknown probability $P_{\Theta_X}(X)$ can be seen as a random variable x that can take its values in the belief interval with some *unknown* density function (pdf) $p(x)$.

Although it is obvious that the knowledge of bounds of the belief interval doesn't suffice to identify the whole shape of the unknown pdf of the random variable $x \equiv P_{\Theta_X}(X)$, Jøsang in 1997 did propose in [1] to make an equivalence (a bijective mapping) between the BBA $m_{\Theta_X}(.)$ and the Beta[18] pdf $p(x; \alpha, \beta)$ defined for real parameters $\alpha > 0$ and $\beta > 0$ by [18]:

$$p(x; \alpha, \beta) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } x \in [0, 1] \\ 0 & \text{for } x \notin [0, 1] \end{cases} \quad (40)$$

where $B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ is the Beta function that allows to normalize the pdf to one, i.e $\int_0^1 p(x; \alpha, \beta)dx = 1$. The Beta pdf $p(x; \alpha, \beta)$ is well-known and attractive in statistics because it can model different shapes of distributions of x on the support[19] $[0, 1]$. The Beta pdf family is a flexible way to model random variables on the interval [0,1]. It is often used to model pdf's for proportions. The expected value of x is $E[x] = \int_0^1 xp(x; \alpha, \beta)dx = \frac{\alpha}{\alpha+\beta}$. When $\alpha = \beta = 1$, the pdf $p(x; \alpha, \beta)$ corresponds to the uniform pdf of x on $[0, 1]$.

The link between $m_{\Theta_X}(.)$ and parameters α and β of the Beta pdf has been proposed in 1997 in a first parameter mapping (α_1, β_1). In [1], the author considers only[20] Beta

[17] because it is based on a "relative dogmatism" factor $\gamma \triangleq m_1(X \cup \bar{X})/m_2(X \cup \bar{X})$ which is also mathematically indeterminate if $K = 0$ because if $m_1(X \cup \bar{X}) = m_1(X \cup \bar{X}) = 0$ then $\gamma = 0/0$ (indeterminacy), see [13] for details.

[18] Beta pdf is a special case of the multinomial Dirichlet pdf.

[19] and on any interval in fact by some transformation of the random variable.

[20] in order to exclude U-shaped Beta pdf's for obscure reasons.

pdf's with $\alpha_1 \geq 1$ and $\beta_1 \geq 1$ chosen as follows[21]:

$$\begin{cases} \alpha_1 = 1 + \frac{b_{\Theta_X}(X)}{u_{\Theta_X}(X)} = \frac{Pl_{\Theta_X}(X)}{Pl_{\Theta_X}(X) - Bel_{\Theta_X}(X)} \\ \beta_1 = 1 + \frac{d_{\Theta_X}(X)}{u_{\Theta_X}(X)} = \frac{Pl_{\Theta_X}(\bar{X})}{Pl_{\Theta_X}(X) - Bel_{\Theta_X}(X)} \end{cases} \tag{41}$$

Later in 2003 and in 2004, the authors did adopt a second parameter mapping (α_2, β_2) in [14], [15], [19] to include the prior information (base-rate) one has (if any) on the cardinalities of X and $\bar{X}$. This second mapping is given by:

$$\begin{cases} \alpha_2 = 2 \cdot a(X|\Theta_X) + \frac{2b_{\Theta_X}(X)}{u_{\Theta_X}(X)} \\ \beta_2 = 2 \cdot (1 - a(X|\Theta_X)) + \frac{2d_{\Theta_X}(X)}{u_{\Theta_X}(X)} \end{cases} \tag{42}$$

where $a(X|\Theta_X)$ and $1 - a(X|\Theta_X) = a(\bar{X}|\Theta_X)$ are the relative atomicities of X and $\bar{X}$ w.r.t the frame Θ_X.

Clearly these two mappings are inconsistent so that at least one of them is wrong, because even if we consider same cardinalities for X and $\bar{X}$ so that $a(X|\Theta_X) = a(\bar{X}|\Theta_X) = 1/2$ then (42) gives us $\alpha_2 = 1 + \frac{2b_{\Theta_X}(X)}{u_{\Theta_X}(X)} \neq \alpha_1$ and $\beta_2 = 1 + \frac{2d_{\Theta_X}(X)}{u_{\Theta_X}(X)} \neq \beta_1$. Note that this second mapping is not bijective[22] so that no strict equivalence between the BBA and Beta pdf can be made. More recently, in a book in preparation [20], p. 16, the author uses another mapping (α_3, β_3) by choosing $\alpha_3 = 1 + r = 1 + \frac{W \cdot b_{\Theta_X}(X)}{u_{\Theta_X}(X)}$ and $\beta_3 = 1 + s = 1 + \frac{W \cdot d_{\Theta_X}(X)}{u_{\Theta_X}(X)}$, where the weight W is a parameter characterizing the a priori one has on X. Clearly this new third mapping is not consistent with the second mapping, and it is consistent with the first mapping only if $W = 1$. So, in SL what is the correct mapping (if any)?

VI. Conclusions

In this paper, we have examined with attention some bases of SL from the fusion standpoint. SL deals with so-called opinions that are basically just simple basic belief assignments defined over the power set of a 2D frame of discernment obtained from the coarsening of a refined frame. Our analysis justifies the doubts on the validity and the interest of SL for combining opinions about propositions expressed in a common 2D frame because at least two of its main bases (its normal model for building opinions, as well as its consensus fusion rule) appear clearly very perfectible. Therefore, we cannot reasonably recommend the Subjective Logic for fusion applications.

Acknowledgements

This work was supported by Grant for State Key Program for Basic Research of China (973) (No. 2013CB329405), National Natural Science Foundation of China (No.61104214, No. 61203222), the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20120201120036), and by project AComIn, grant 316087, funded by the FP7 Capacity Programme.

[21] assuming $u_{\Theta_X}(X) = m_{\Theta_X}(X \cup \bar{X}) > 0$.

[22] because we cannot compute the three free parameters, say $(b_{\Theta_X}(X), d_{\Theta_X}(X), a(X|\Theta_X))$, from the knowledge of the two parameters (α_2, β_2) only.

References

[1] A. Jøsang, Artificial reasoning with subjective logic, 2nd Workshop on commonsense reasoning, Australian Comp. Soc., Dec. 1997.

[2] G. Shafer, A Mathematical theory of evidence, Princeton University Press, Princeton, NJ, U.S.A., 1976.

[3] A. Jøsang, A Logic for uncertain probabilities, Int. J. of Uncert. Fuzz. and Knowledge-Based Syst., Vol. 9(3), pp. 279–311, June 2001.

[4] A. Jøsang, D. McAnally, Multiplication and comultiplication of beliefs, IJAR, Vol. 38, pp.19–51, 2005.

[5] J. Dezert, P. Wang, A. Tchamova, On the validity of Dempster-Shafer theory, Proc. of Fusion 2012 Conf., Singapore, July 9–12, 2012.

[6] A. Tchamova, J. Dezert, On the behavior of Dempster's rule of combination and the foundations of Dempster-Shafer theory, 6th IEEE Int. Conf. Intell. Syst. (IS 12), pp. 108–113, Sofia, Bulgaria, 2012.

[7] J. Dezert, A. Tchamova A., On the validity of Dempster's fusion rule and its interpretation as a generalization of Bayesian fusion rule, Int. J. of Intell. Sys., Vol. 29, pp. 223–252, 2014.

[8] A. Jøsang, V.A. Bondi, Legal reasoning with subjective logic, Artificial Intelligence and Law, Vol. 8(4), pp. 289–315, 2000.

[9] A. Jøsang, The consensus operator for combining beliefs, Artif. Intell. J., 142 (1-2), pp. 157–170, Oct. 2002.

[10] P. Smets, R. Kennes, The transferable belief model, Artif. Intell. J., Vol. 66, pp. 191–243, 1994.

[11] F. Smarandache, J. Dezert (Ed.), Advances and applications of DSmT for information fusion, Vol. 3, Amer. Res. Press, USA, 2009.

[12] J. Dezert, Autonomous navigation with uncertain reference points using the PDAF, Chap. 9 in Multitarget-Multisensor Tracking: Applications and Advances, Vol. 2, Y. Bar-Shalom Editor, Artech House, 1991.

[13] A. Jøsang, Subjective evidential reasoning, Proc. the 9th IPMU Conf., Annecy, France, 1–5 July, 2002.

[14] A. Jøsang et al., Strategies for combining conflicting dogmatic beliefs, Proc. of Fusion 2003 Conf., Cairns, Australia, 8-11 July, 2003.

[15] A. Jøsang, T. Grandison, Conditional inference in subjective logic, Proc. of Fusion 2003 Conf., Cairns, Australia, 8-11 July, 2003.

[16] C.E. Shannon, A mathematical theory of communication, Bell Syst. Techn. Journal, Vol. 27, pp. 379–423, July, October, 1948.

[17] A. Feinstein, Foundations of information theory, McGraw-Hill, 1958.

[18] M. Abramowitz, I.A. Stegun (Eds.), Handbook of mathematical functions, 9th printing, New York, Dover, pp. 944–945, 1972.

[19] A. Jøsang, D. McAnally, Multiplication and comultiplication of beliefs, IJAR, Vol. 38(1), pp.19–51, 2004.

[20] A. Jøsang, Subjective Logic, (book draft), 2013. http://folk.uio.no/josang/papers/subjective_logic.pdf